Reply to "Melnikov function and homoclinic chaos induced by weak perturbations"

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For the case of weak perturbations, our theory is shown to reproduce *exactly* the results of Simiu and Frey [Phys. Rev. E **48**, 3185 (1993)]. In the presence of weak noise, the two approaches yield different results. This can be traced to the neglect of diffusion effects in the Simiu-Frey theory; the inclusion of these effects, via an ensemble-averaged redefinition of the Melnikov function, leads to an additional term involving the noise variance.

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The authors Simiu and Frey (hereafter referred to as SF) of the preceding comment [1] have one basic objection to our work: our redefined "generalized Melnikov function," introduced in our first paper (hereafter referred to as BSJ) [2], is inconsistent with the original mathematical definition—it is not "geometrically correct." We have admitted, from the outset, that by raising the issue of "homoclinic tangency in the mean" there may be questions concerning the validity of our extension of the Melnikov approach, showing transverse intersections and implying horseshoes, in the presence of noise (in this case meaningful, i.e., measurable, quantities are considered as suitably defined ensemble averages). Nevertheless, our conjecture that one could carry over deterministic techniques to the macroscopic quantities that describe the position and velocity variables in the ensemble-averaged system is one that is firmly rooted in the stochastic repertoire. We must also point out that the whole concept of homoclinic tangency in the presence of noise remains undefined. We have approached the problem from a physicist's standpoint: writing down a stochastic analog of the Melnikov function (through a redefined "unperturbed" system) and then making contact with the usual definition of homoclinic tangency in the *ensemble-averaged* framework. Our results reduce to the well-known result (also the SF result) for the nonnoisy case. For the case of nonzero noise, the SF calculation sheds very little light on the fundamental issue enunciated above; we discuss these points in greater detail below.

In BSJ, we consider the system

$$\ddot{x}+f(x)=F(t)+Q\sin[\omega(t-t_0)]-k\dot{x}\ , \tag{1}$$

where $F(t)$ is Gaussian δ-function-correlated noise with mean m and variance σ^2. We assume that (for $k=0=Q$) the "weak" (this term is quantified in Sec. III of BSJ) noise leads to trajectories that fluctuate in position and velocity about the deterministic separatrix. In BSJ, we considered the $m=0$ case throughout, the exception being the digression leading to Eq. (40). However, we contend that there should be a diffusion contribution as well. Such a contribution can be calculated within a Fokker-Planck framework which is what our original work was about. In fact, let us assume that, in the presence of noise, the Melnikov function is written as the ensemble-averaged quantity,

$$\Delta_F=\Delta_0+\Delta_c\ , \tag{2}$$

where

$$\Delta_0=m\int dt\,\dot{x}_s(t)-k\int dt\,\dot{x}_s^2(t)+Q\int dt\,\dot{x}_s(t)\sin[\omega(t-t_0)] \tag{3}$$

is the Melnikov function written down by SF. As noted above, this function involves a contribution due to the mean value m of the noise (a contribution from the drift part of the equivalent Fokker-Planck equation). The correction term Δ_c contains a diffusion contribution:

$$\Delta_c\equiv k\int dt\langle\delta\dot{x}^2(t)\rangle\ . \tag{4}$$

To obtain this correction resulting from higher-order (in the noise) perturbations of the separatrix [recall that in this picture, the first-order correction $\langle\delta x(t)\rangle=0$] we write down the perturbed equation

$$\ddot{x}+f(x)=\zeta(t)\ , \tag{5}$$

where we have set $\zeta(t)=F(t)-m$, and carry out the expansion

$$x(t)=x_s(t)+\delta x(t), \tag{6}$$

which is to lowest order $[0(\sqrt{\sigma_1^2})]$ in the noise. The correction term Δ_c then turns out to be $O(\sigma_1^2)$, however, it is still of the same order in the Melnikov perturbation theory, i.e., $O(k,Q,m)$. The net result is that Eq. (36) of BSJ (which was originally derived for $m=0$) should now be replaced by

$$\frac{Q}{k}=\left[\frac{Q}{k}\right]_0+B\sigma_1^2-\frac{m}{k}G\ , \qquad (7)$$

where

$$G\equiv\frac{\int \dot{x}_s(t)dt}{\int \dot{x}_s(t)\sin[\omega(t-t_0)]dt}$$

and B is defined in BSJ [Eq. (38)]. This equation also takes the place of Eq. (40) of BSJ.

In the absence of noise, but with additional perturbations of the form $\gamma_1\cos\omega_1 t$ (as suggested by SF), the second and third terms on the right-hand side (rhs) of (7) are replaced by terms similar to the third term on the rhs of (3); our theory reproduces, for this case, the SF results with the presence of additional periodic perturbations, indeed lowering the homoclinic threshold. The situation is far more complicated in the presence of noise. For Gaussian noise, the Fokker-Planck representation of the dynamics as a diffusion process is exact. No higher-order terms occur in the Fokker-Planck equation, but the effects of diffusion are not zero either. This leads, in turn, to the second term on the rhs of (7). It is the interplay between the second and third terms on the rhs of (7) that determines whether the homoclinic threshold is elevated or depressed. Note that in our original paper, we considered only the $m=0$ case; for this case the threshold can only be elevated.

Before going further, it is instructive to make a few comments about the numerical simulation of noisy nonlinear dynamic systems. SF use the Shinozuka algorithm that represents the noise as an infinite Fourier decomposition. The Shinozuka algorithm has been compared with other algorithms in Ref. [3]. In theory, a very large number of terms must be retained in the Fourier decomposition of the noise, to approach real Gaussian statistics; the Gaussian distribution is obtained only in the $N\to\infty$ limit. In practice, however, about 1000 terms will usually suffice [4] to generate the tails of the Gaussian distribution with about 5% accuracy, and guarantee ergodicity in the averaging process. The SF simulation in which only 15 terms are retained falls far short of these requirements. They are, in effect, simulating not noise, but the effect of including 15 periodic perturbations with random frequencies and phases. They point out (rightly, and in agreement with our results as indicated above) that this depresses the homoclinic threshold. However, they then seem to imply that this simulation may reproduce (qualitatively at least) the effects of noise. This argument is somewhat nebulous. To begin with, the whole notion of considering homoclinic crossing in the ensemble-averaged framework means that one should consider thousands of realizations of the noise (corresponding to different random number seeds) and take the ensemble average of all these realizations. Recent algorithms [5] are ideally suited to carry out these simulations and have, in fact, been used in Ref. [2] to generate Figs. 6–12. The SF simulation (even if they employed additional terms in their spectral representation) corresponds to only a *single* realization of the noise. There is, however, a far deeper issue. It is unclear, at this point, how one takes into account diffusion effects if one introduces into the nonlinear dynamics (1) a Fourier representation of the noise, as employed by SF; the passage to a diffusion involving a probability density function is not clear in this case. Clearly, this is a problem which, in its own right, merits further consideration. On the other hand, the Fokker-Planck approach for real Gaussian noise, defined simply via its first two moments and *not* via the spectral representation, is, as mentioned earlier, exact, and results obtained via its application have been successfully tested, for a large class of applications, by direct simulation of the corresponding stochastic differential equations using any of the algorithms of Ref. [5]. We contend therefore that, pending further investigation into the questions raised above (and in BSJ), our calculation is far more general than SF. We reiterate that our work is consistent with the original perturbation expansion of Melnikov; the ensemble-averaged correction Δ_c is of the same order of perturbation theory as the original Melnikov function (in the absence of noise). The factor σ_1^2 enters because we have introduced a second expansion in a different variable (the noise) in order to calculate noise-induced perturbations to the separatrix solution.

We would like to make a couple of comments regarding the multiplicative noise case [6]. For this case, we have considered the system

$$\ddot{x}+\beta(t)f(x)=Q\sin(\omega t)-k\dot{x}\ , \qquad (8)$$

where $\beta(t)$ is now Gaussian δ-function-correlated noise having a *nonzero* mean. If we follow the procedure of SF, the way to treat this problem is to solve the above equation with the rhs set equal to zero, and use this solution in the original definition of Melnikov. To the best of *our* knowledge, there exists at present no known mathematical technique for accomplishing this. Our procedure would be the only reasonable way to treat this problem. Clearly, for the multiplicative noise case, the homoclinic threshold will be raised or lowered depending on the sign of the mean value m of the noise (the effect of multiplicative noise is to modulate, on average, the potential barrier height).

In conclusion, we disagree with the SF contentions. For them to claim that conventional Melnikov theory can be straightforwardly applied to a nonlinear stochastic system and then to ignore diffusion effects is contrary to the most basic principles of nonequilibrium statistical mechanics. By merely rewriting the function Δ_0 (which is already contained in our theory), SF have not shed any new light on the problem; rather, they have chosen to present us with a calculation which is simply a nonrigorous extension of well-known earlier work done by Wiggins [7] on the two-frequency-forced Duffing problem. For the reasons mentioned in the preceding paragraph, the somewhat limited example that they have simulated cannot be used as a basis for claiming that their theory is correct unless they can *prove* that there are no diffusion effects; they cannot do so with their current approach.

We would like to reiterate in closing that there do indeed exist some very profound questions concerning the representation of noise via its spectral decomposition and its integration into the conventional approach based on a diffusion equation for the probability density function of the dependent variable, as well as the broader issue of homoclinic tangency in an ensemble-averaged sense. At present, a rigorous proof of the existence of Smale horseshoes exists only for the two-frequency-forced case [7]. We have tried to provide a starting point for the extension of the concept of homoclinic tangency to noisy systems by introducing the notion of homoclinic tangency in the mean. For both the additive and multiplicative noise cases, our procedure has been to introduce a Melnikov function (following the original definition) for one member of the ensemble (i.e., one realization of the noise) and then average over the ensemble.

[1] E. Simiu and M. Frey, preceding paper, Phys. Rev. E **48**, 3185 (1993).
[2] A. Bulsara, W. Schieve, and E. Jacobs, Phys. Rev. A **41**, 668 (1990).
[3] K. Billah and M. Shinozuka, Phys. Rev. A **42**, 7492 (1990).
[4] J. Yang, J. Sound Vib. **26**, 417 (1973).
[5] J. Sancho *et al.*, Phys. Rev. A **26**, 1589 (1982); R. Fox *et al.*, *ibid.* **38**, 5938 (1988); R. Mannella and V. Palleschi, *ibid.* **40**, 3381 (1989).
[6] W. Schieve and A. Bulsara, Phys. Rev. A **41**, 1172 (1990).
[7] S. Wiggins, Phys. Lett. **124**, 138 (1987); *Global Bifurcations and Chaos* (Springer-Verlag, Berlin, 1988), p. 463, Theorem 4.2.2.